Graphs Induced by Complemented Elements of Commutative Rings

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8. $(a + b)c = ac + bc$.

Examples

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$\mathbb{N}$ is **not** a commutative ring because any element does not have an additive inverse.

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$\mathbb{R}$ is a commutative ring with the usual addition and multiplication.

Regular elements and zero divisors

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Nonzero elements x and y such that $xy = 0$ are called *zero divisors*. Denote the set of zero divisors of R as $Z(R)$.

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Example

In $\mathbb{Z}_6$, $2 \times 3 = 0 \mod 6$, so 2 and 3 are both zero divisors. However, 5 is not a zero divisor, so it is considered regular.

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Graphs

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A graph $G(V, E)$ consists of a set V of vertices and a set E of edges such that each edge connects two of elements of V.

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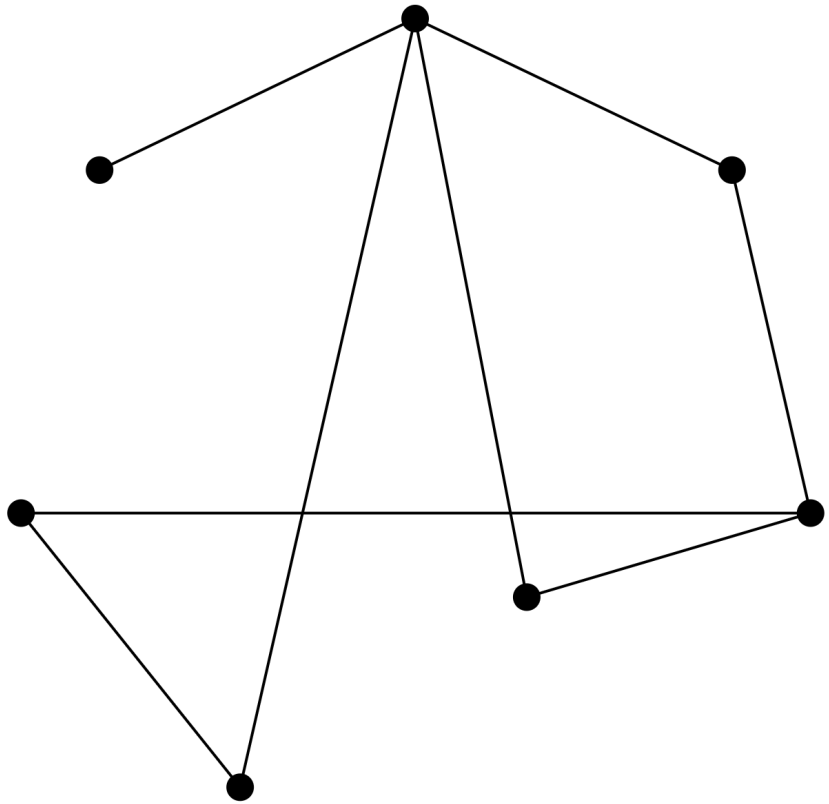


Figure 1: Graph

Complete bipartite graph

Definition

A complete bipartite graph $K_{m,n}$ is a graph whose vertices are split into two groups of sizes m and n. Every vertex in one group is connected to every vertex in the other group, and no two vertices in the same group are connected.

Zero-Divisor Graphs

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History

Refined in [?] to only include $Z(R)$ as vertices.

Examples

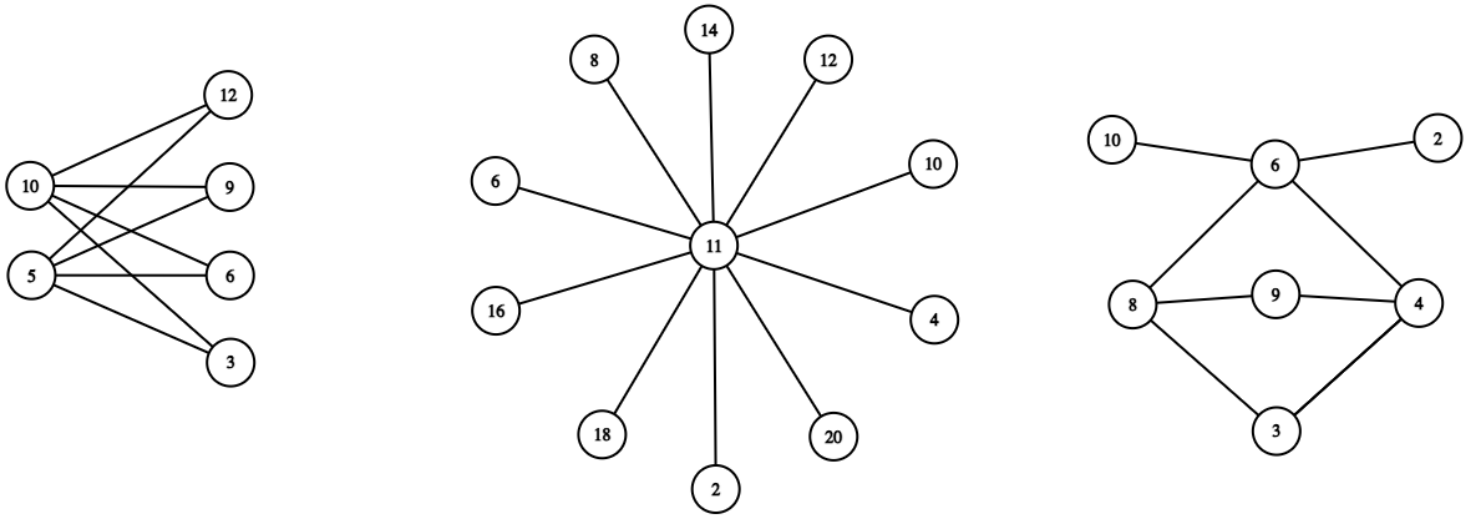


Figure 2: Zero Divisor Graphs of $\mathbb{Z}_{15}$, $\mathbb{Z}_{22}$, and $\mathbb{Z}_{12}$

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Complemented elements

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An element $x \in R$ is called *complemented* if there exists $y \in R$ such that $xy = 0$ and $x + y \in reg(R)$. Denote the set of complemented elements of R as $\mathcal{C}(R)$.

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Example

In $\mathbb{Z}_{15}, 3 \times 5 = 0$ and $3 + 5 = 8 \in reg(\mathbb{Z}_{15})$, so $3, 5 \in \mathcal{C}(\mathbb{Z}_{15})$

Motivation

When is $Z(R)$ **not** closed under addition?

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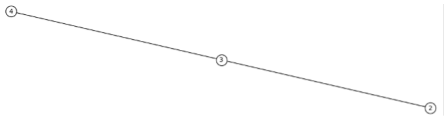
How do the elements interact with each other?

Our Construction

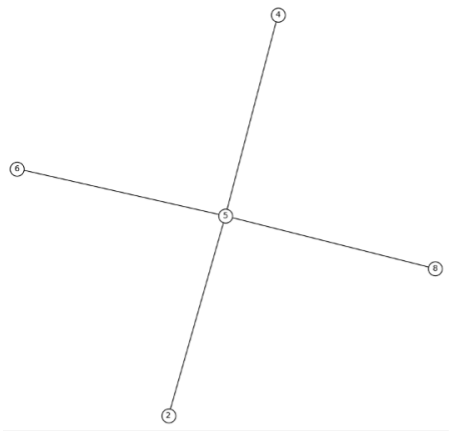
Definition

We denote $\Gamma_C(R)$ as the graph induced by the complemented elements in R. The vertices in $\Gamma_C(R)$ are precisely the complemented elements in R and two vertices x, y are adjacent if and only if $xy = 0$ and $x + y \in reg(R)$.

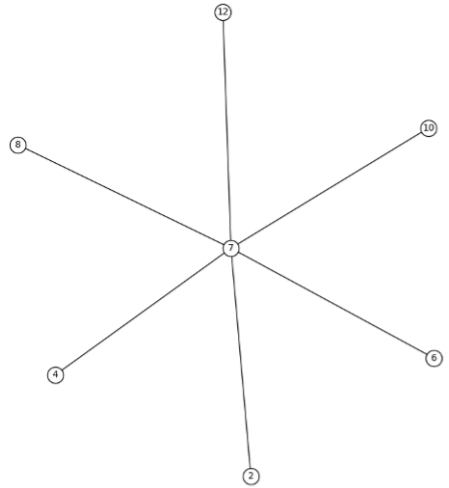
Examples



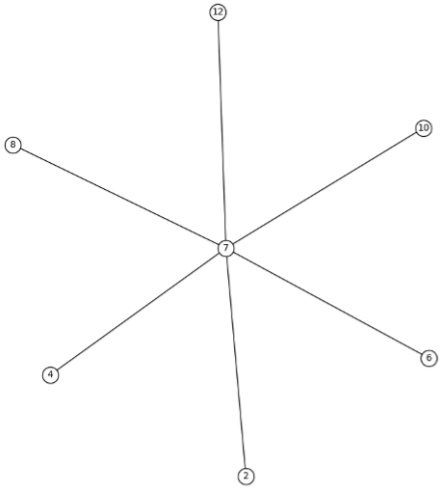
$\Gamma_C(\mathbb{Z}_6)$



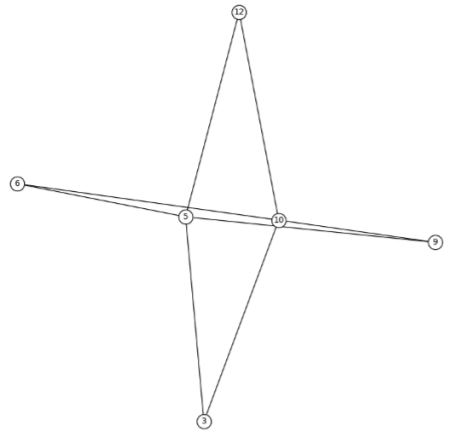
$\Gamma_C(\mathbb{Z}_{10})$



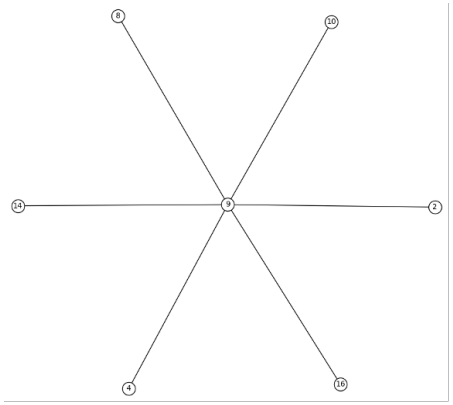
$\Gamma_C(\mathbb{Z}_{12})$



$\Gamma_C(\mathbb{Z}_{14})$

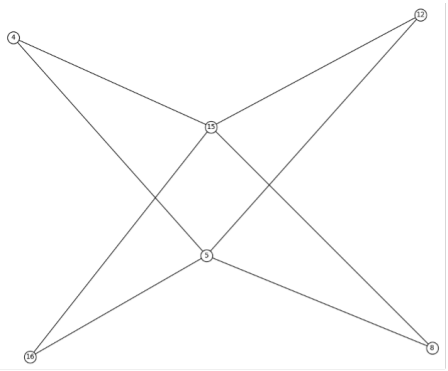


$\Gamma_C(\mathbb{Z}_{15})$

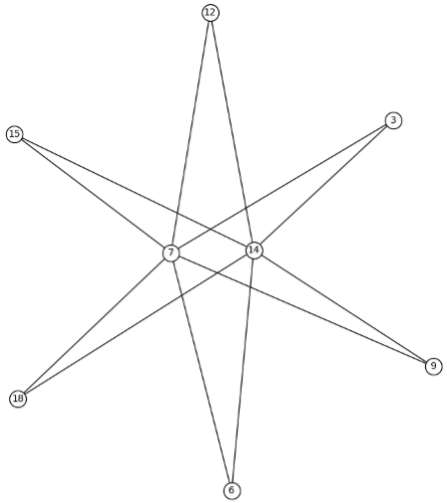


$\Gamma_C(\mathbb{Z}_{18})$

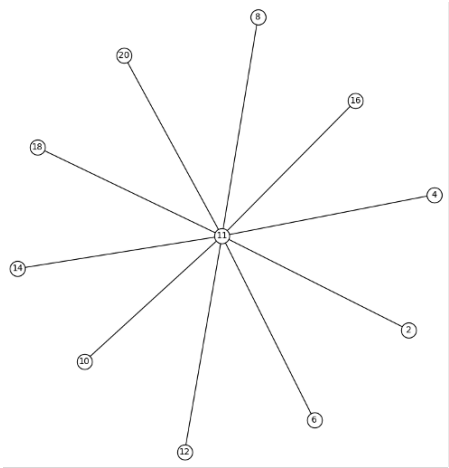
Examples (Cont'd)



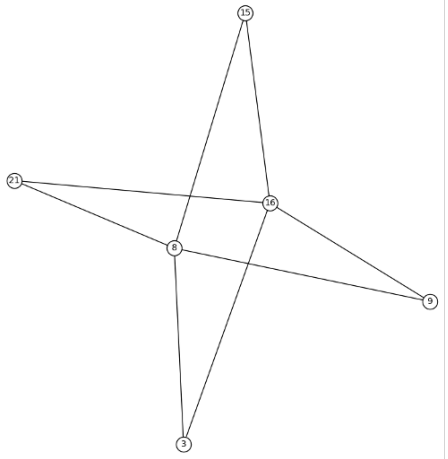
$\Gamma_C(\mathbb{Z}_{20})$



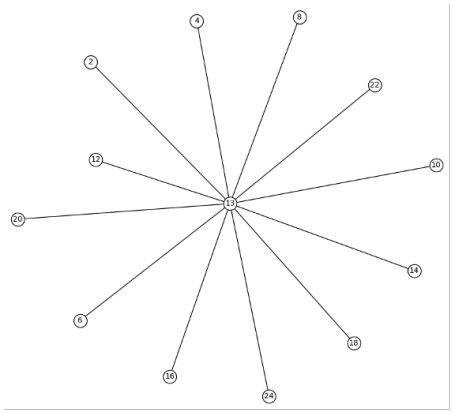
$\Gamma_C(\mathbb{Z}_{21})$



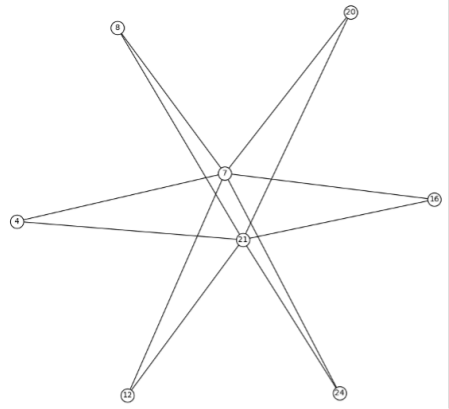
$\Gamma_C(\mathbb{Z}_{22})$



$\Gamma_C(\mathbb{Z}_{24})$



$\Gamma_C(\mathbb{Z}_{26})$



$\Gamma_C(\mathbb{Z}_{28})$

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Theorem

$$\Gamma_C(\mathbb{Z}_n) \cong \bigsqcup_{\emptyset \neq S,T \subset \{p_1,\ldots,p_k\}, S \cup T = \{p_1,\ldots,p_k\}} K_{|E_S|,|E_T|}$$

where $\{S, T\}$ *is unordered, and*

$$|E_S| = \varphi(\prod_{p_i \in T} p_i^{e_i}), |E_T| = \varphi(\prod_{p_j \in S} p_j^{e_j})$$

and

$$|E(K_{|E_S|,|E_T|})| = \varphi(n)$$

where φ *denotes the Euler's totient function.*

Proof Sketch

Proof.

- We can split $\{p_1, \ldots, p_k\}$ into two groups S and T in 2^k ways if order matters.
- Since the pair $\{S, T\}$ is the same as $\{T, S\}$, we divide by 2 to get 2^{k-1} unordered pairs.
- One of these pairs is $\{\emptyset, \{p_1, \ldots, p_k\}\}$, which doesn't correspond to any complemented elements, so we subtract it.
- That leaves $2^{k-1} - 1$ pairs, each giving one connected component that is a complete bipartite graph.
- Adding up all the vertices across every component gives us $(2^{k-1} - 1)\varphi(n)$ total vertices.

□

Characterization of Planarity

Theorem (Kuratowski,[?])

A finite graph $G(V, E)$ is planar if and only if it does not contain a subgraph that is homemoprhic to K_5 or $K_{3,3}$.

Lemma

$\varphi(n) \leq 2$ *if and only if* $n = 1, 2, 3, 6$.

Theorem

$\Gamma_C(\mathbb{Z}_n)$ *is planar if and only if*
$n \in \{k^{e_1}p^{e_2} \mid k = 1, 2, 3, 6, p$ *is prime or* $p = 1\}$.

Proof.

First show by Kuratowski's theorem that each bipartite component $K_{n,m}$ is planar if and only if $\min\{n, m\} \leq 2$.
Then apply the above lemma and the theorem follows. □

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Conclusion

- We defined a new graph $\Gamma_C(\mathbb{Z}_n)$ built from complemented elements of $\mathbb{Z}_n$.
- We characterized exactly which elements are complemented: an element must be divisible by the full prime power $p_i^{e_i}$ or not at all.
- We proved $\Gamma_C(\mathbb{Z}_n)$ decomposes as a disjoint union of $2^{k-1}-1$ complete bipartite graphs, where k is the number of distinct prime factors of n.
- The total vertex count is $(2^{k-1}-1)\varphi(n)$.
- We also characterized when the graph is planar.

Future Directions

- **Other rings:** Can we describe $\Gamma_C(R)$ for other families of rings, like polynomial rings $\mathbb{Z}_n[x]/(f(x))$ or direct products of rings?
- **Relationship to zero-divisor graph:** How does $\Gamma_C(\mathbb{Z}_n)$ sit inside the classical zero-divisor graph $\Gamma(\mathbb{Z}_n)$? It is a subgraph, what structural properties does it inherit or lose?
- **Graph invariants:** Compute the chromatic number, clique number, and independence number for $\Gamma_C(\mathbb{Z}_n)$.
- **Classification:** Classify the family of $\Gamma_C(R)$ for all commutative rings R up to isomorphism.

References