Stony Brook University

SUNY The State University of New York

On Discrete Curve Shortening Flows

AMS 545 Computational Geometry

Jiaxuan Chen

May 5th, 2026

Smooth Curve Shortening Flow



Let γ be a smooth curve in $\mathbb{R}^2$ parametrized by t.

Definition

Denote the unit tangent vector by $T(t)$, the unit normal vector by $\mathrm{n}(t)$. The *curvature* of γ at time t, denoted $\kappa(t)$, is the radius of the *osculating circle* at t to γ, and is given by

$$\kappa \mathrm{n} = T'.$$

Definition

A *curve shortening flow* is a family of curves $\gamma(\cdot, t)$ given by a solution to the PDE

$$\frac{\partial \gamma}{\partial t} = \kappa \mathrm{n}$$

with initial condition $\gamma(\cdot, 0) = \gamma$ where the first parameter parametrizes each curve.

Properties of the Smooth CSF



Theorem ([?])

Let $L(t)$ be length of $\gamma(\cdot, t)$, let $A(t)$ be the enclosed area (if defined)

$$\frac{dL}{dt} < 0.$$

$$\frac{dA}{dt}(t) = -2\pi < 0.$$

Theorem (Gage-Hamilton [?])

The CSF preserves convexity and shrinks any closed convex curve to a point. In particular it shrinks to a circle.

Theorem (Grayson [?])

The CSF starting at any closed embedded curve becomes convex before the region inside the curve vanishes.

The discrete generalization



Definition (Discrete curve)

A *discrete curve* is a sequence of vertices $v_1, ..., v_n \in \mathbb{R}^2$ joined by line segments $v_{i+1} - v_i$ in order. Let l_i be side length, θ_i be inner angle.

Note that κ is undefined everywhere, since the second derivative of a linear function vanishes, then the normal also vanishes. We need to define n and κ for vertices of a discrete curve. There are different ways to do this.

Definition

In [?], $T_i' = (v_{i+1} - v_i) + (v_{i-1} - v_i)$.
In [?], n_i obtained by rotation, $\kappa_i = \frac{\theta_i}{l_i}$.
In [?], n_i determined by angle bisector, $\kappa = \pi - \theta_i$.
A lot more discrete models proposed.

Properties of the Discrete CSF



Different have different proven properties, although many aspects have not been fully characterized.

Theorem ([?])

A simple closed discrete curve has decreasing area and length, with area shrinking in finite time with speed -2π. It is conjectured that the asymptotic shape is an equiangular $N-$gon.

Theorem ([?])

Every initial closed curve (not necessarily simple), shrinks to a point, and converges asymptotically to an affine transformation of a regular $n-$gon.

Theorem ([?])

A triangle flows to a point if and only if it is an isosceles triangle with top angle $\leq \frac{\pi}{3}$. Otherwise, flows to a line. For convex polygon, proven to be convexity-preserving.

Computational Application



- The motivation of discrete CSF is essentially *computational*.
- We propose an algorithm to visualize or to approximate a smooth CSF with a computer, we need to take finitely many vertices on the smooth curve.
- We need to find a way to compute the discrete CSF to achieve this.
- However, it is still an open problem whether the discrete CSF is asymptotically an accurate approximation of the smooth CSF.

Bibliography

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Project for Research Science Institute, Department of Mathematics, Massachusetts Institute of Technology.