Graphs Induced by Complemented Elements of Commutative Rings

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Abstract

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- To characterize the relationship between zero divisors of a commutative ring, we introduce the notion of complemented elements.
- In analog to the zero-divisor graph of a commutative ring, we define a new graph $\Gamma_C(R)$ induced by complemented elements of a commutative ring R.
- In particular, we examine graph-theoretic properties of the graph $\Gamma_C(\mathbb{Z}_n)$.
- We have shown that this is a union of disjoint complete bipartite graphs.
- We characterized the number of vertices and edges of this graph
- We also characterized when this graph is planar.

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Commutative Rings

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Definition

A commutative ring R is a set equipped with addition and multiplication such that the following properties hold for all $a, b, c \in R$:



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1. $(a + b) + c = a + (b + c)$

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3. There exists some element $0 \in R$ such that for all $a \in R$ we have $0 + a = a$.

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4. For each $a \in R$ there exists some element $-a$ such that $a + (-a) = 0$.

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5. $(ab)c = a(bc)$.

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6. There exists some element $1 \in R$ such that for each $a \in R$ we have $1 \cdot a = a$.



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7. $ab = ba$.
8. $(a + b)c = ac + bc$.



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Example

$\mathbb{N}$ is **not** a commutative ring because any element does not have an additive inverse.



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Example

$\mathbb{Z}_n = \{0, \ldots, n-1\}$ is a commutative ring with addition and multiplication modulo n.

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Example

$\mathbb{Q}$ is a commutative ring with the usual addition and multiplication.



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Example

$\mathbb{Q}$ is a commutative ring with the usual addition and multiplication.

Example

$\mathbb{R}$ is a commutative ring with the usual addition and multiplication.



Regular elements and zero divisors

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Definition

Nonzero elements x and y such that $xy = 0$ are called *zero divisors*. Denote the set of zero divisors of R as $Z(R)$.



Regular elements and zero divisors

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Nonzero elements x and y such that $xy = 0$ are called *zero divisors*. Denote the set of zero divisors of R as $Z(R)$.

Definition

Elements that are not zero divisors are called *regular*. Denote the set of regular elements of R as $reg(R)$.



Regular elements and zero divisors

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Elements that are not zero divisors are called *regular*. Denote the set of regular elements of R as $reg(R)$.

Example

In $\mathbb{Z}_6$, $2 \times 3 = 0 \mod 6$, so 2 and 3 are both zero divisors. However, 5 is not a zero divisor, so it is considered regular.



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Definition

A graph $G(V, E)$ consists of a set V of vertices and a set E of edges such that each edge connects two of elements of V.

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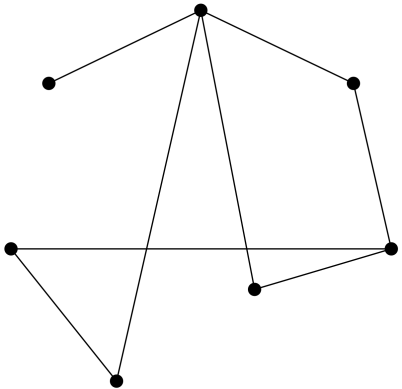


Figure: A graph



Complete bipartite graph

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Definition

A complete bipartite graph $K_{m,n}$ is a graph whose vertices are split into two groups of sizes m and n. Every vertex in one group is connected to every vertex in the other group, and no two vertices in the same group are connected.



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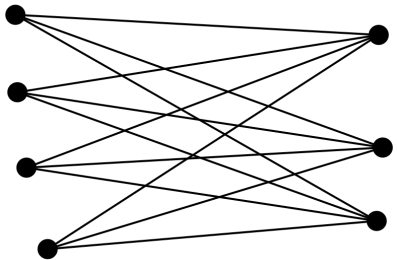


Figure: $K_{3,4}$



Zero-Divisor Graphs

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A zero-divisor graph is an graph where the vertices represent all the zero-divisors of a commutative ring. Two distinct vertices, x and y, are connected by an edge if and only if their product equals zero ($xy = 0$).



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History

Originally introduced in Beck 1986 concerning colorings. Every element of R were vertices.



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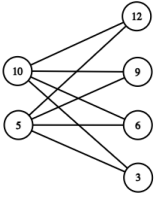
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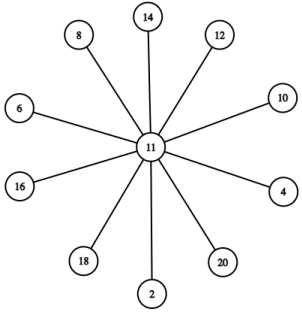
History

Refined in Anderson and Livingston 1999 to only include $Z(R)$ as vertices.

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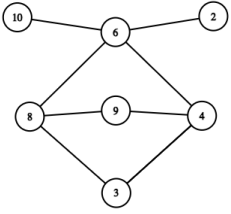


Figure: Zero Divisor Graphs of $\mathbb{Z}_{15}$, $\mathbb{Z}_{22}$, and $\mathbb{Z}_{12}$



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Complemented elements

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Definition

An element $x \in R$ is called *complemented* if there exists $y \in R$ such that $xy = 0$ and $x + y \in reg(R)$. Denote the set of complemented elements of R as $\mathcal{C}(R)$.



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In $\mathbb{Z}_6, 2 \times 3 = 0$ and $2 + 3 = 5 \in reg(\mathbb{Z}_6)$ so $2, 3 \in \mathcal{C}(\mathbb{Z}_6)$



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Example

In $\mathbb{Z}_{15}, 3 \times 5 = 0$ and $3 + 5 = 8 \in reg(\mathbb{Z}_{15})$, so $3, 5 \in \mathcal{C}(\mathbb{Z}_{15})$

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When is $Z(R)$ **not** closed under addition?

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When is $Z(R)$ **not** closed under addition?

How do the elements interact with each other?

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Definition

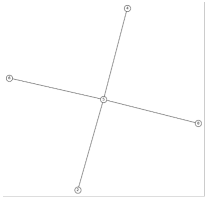
We denote $\Gamma_C(R)$ as the graph induced by the complemented elements in R. The vertices in $\Gamma_C(R)$ are precisely the complemented elements in R and two vertices x, y are adjacent if and only if $xy = 0$ and $x + y \in reg(R)$.

Examples

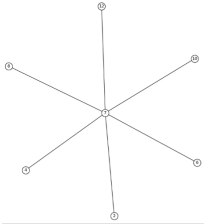
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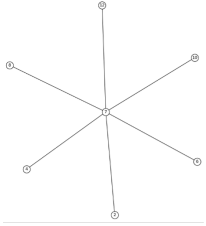
$\Gamma_C(\mathbb{Z}_6)$



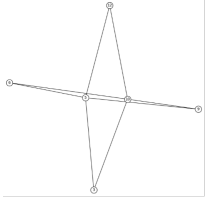
$\Gamma_C(\mathbb{Z}_{10})$



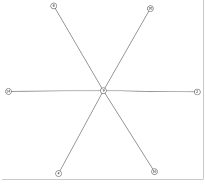
$\Gamma_C(\mathbb{Z}_{12})$



$\Gamma_C(\mathbb{Z}_{14})$



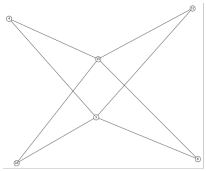
$\Gamma_C(\mathbb{Z}_{15})$



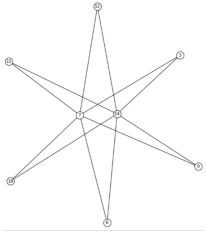
$\Gamma_C(\mathbb{Z}_{18})$

Examples (Cont'd)

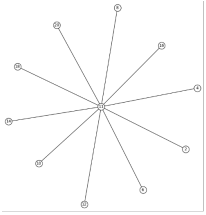
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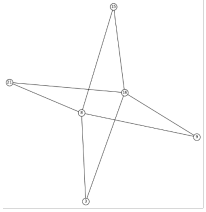
$\Gamma_C(\mathbb{Z}_{20})$



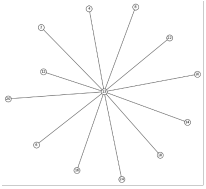
$\Gamma_C(\mathbb{Z}_{21})$



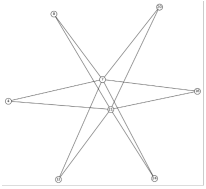
$\Gamma_C(\mathbb{Z}_{22})$



$\Gamma_C(\mathbb{Z}_{24})$



$\Gamma_C(\mathbb{Z}_{26})$



$\Gamma_C(\mathbb{Z}_{28})$



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Definition

On $\mathbb{Z}_n \setminus \{0\}$, define a relation $\mathcal{R}$ in the following way. For $a, b \in \mathbb{Z}_n \setminus \{0\}$ we have $a\mathcal{R}b$ if and only if for all p_i,

$$p_i \mid a \text{ if and only if } p_i \mid b.$$

Lemma (Lemma 1)

Let $E_S := [x]_{\mathcal{R}} \in \mathcal{C}(\mathbb{Z}_n)/\mathcal{R}$ where $S = \{p_i : p_i \mid x\}$. Therefore,

$$\mathcal{C}(\mathbb{Z}_n) = \bigsqcup_{\emptyset \neq S \subsetneq \{p_1, \ldots, p_k\}} E_S$$

which forms a partition of the set of complemented elements.



Characterizing connectivity using the partition

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Lemma (Lemma 2)

Two vertices $a, b \in V(\Gamma_C(\mathbb{Z}_n))$ where $a \in E_S, b \in E_T$ are adjacent only if $S^C = T$ where $S^C = \{p_1, ..., p_k\} \setminus S$.

Lemma (Lemma 3)

The set of complemented elements of $\mathbb{Z}_n$ is

$$\mathcal{C}(\mathbb{Z}_n) = \{x \in Z(\mathbb{Z}_n) : p_i \mid x \text{ if and only if } p_i^{e_i} \mid x, \text{ for all } 1 \leq i \leq k\}.$$

Theorem

In the graph $\Gamma_C(\mathbb{Z}_n)$, for $E_S, E_T \in \mathcal{C}(\mathbb{Z}_n)/\mathcal{R}$ we have

$$x \sim y \text{ for all } x \in E_S, y \in E_T, \text{ if and only if } S^C = T.$$

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Theorem

$$\Gamma_C(\mathbb{Z}_n) \cong \bigsqcup_{\emptyset \neq S,T \subset \{p_1,\ldots,p_k\}, S \cup T = \{p_1,\ldots,p_k\}} K_{|E_S|,|E_T|}$$

where $\{S, T\}$ *is unordered, and*

$$|E_S| = \varphi(\prod_{p_i \in T} p_i^{e_i}), |E_T| = \varphi(\prod_{p_j \in S} p_j^{e_j})$$

and

$$|E(K_{|E_S|,|E_T|})| = \varphi(n)$$

where φ *denotes the Euler's totient function.*

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Proof.

For all $x \in Z(\mathbb{Z}_n)$, $x \in E_S$ if and only if $p_i^{e_i} \mid x$ for all $p_i \in S$ and $p_i \nmid x$. Thus, $x \in E_S$ is a multiple of

$$y = \prod_{p_i \in S} p_i^{e_i}.$$

In particular, $x = ky$ for $1 \leq k \leq n/y$ such that $(x, n/y) = (k, n/y) = 1$. By Euler's totient theorem,

$$|E_S| = \#k = \varphi(\prod_{p_j \in S^C} p_j^{e_j}).$$

The proof for $|E_T|$ is identical. By multiplicity of Euler's totient function,

$$|V(K_{|E_S|,|E_T|})| = |E_S||E_T| = \varphi(\prod_{p_i \in T} p_i^{e_i})\varphi(\prod_{p_j \in S} p_j^{e_j}) = \varphi(n)$$



Number of vertices of the graph

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Corollary

The graph $\Gamma_C(\mathbb{Z}_n)$ has $2^{k-1}-1$ connected components. Thus, $|V(\Gamma_C(\mathbb{Z}_n))| = (2^{k-1}-1)\varphi(n)$.

Proof.

- We can split $\{p_1, \ldots, p_k\}$ into two groups S and T in 2^k ways if order matters. Since the pair $\{S, T\}$ is the same as $\{T, S\}$, we divide by 2 to get 2^{k-1} unordered pairs.
- One of these pairs is $\{\emptyset, \{p_1, \ldots, p_k\}\}$, which doesn't correspond to any complemented elements, so we subtract it.
- That leaves $2^{k-1}-1$ pairs, each giving one connected component that is a complete bipartite graph.
- Adding up all the vertices across every component gives us $(2^{k-1}-1)\varphi(n)$ total vertices.

Characterization of Planarity

Theorem (Kuratowski,Kuratowski 1930)

A finite graph $G(V, E)$ is planar if and only if it does not contain a subgraph that is homemoprhic to K_5 or $K_{3,3}$.

Lemma

$\varphi(n) \leq 2$ *if and only if* $n = 1, 2, 3, 6$.

Theorem

$\Gamma_C(\mathbb{Z}_n)$ *is planar if and only if* $n \in \{k^{e_1}p^{e_2} \mid k = 1, 2, 3, 6, p \text{ is prime or } p = 1\}$.

Proof.

First show by Kuratowski's theorem that each bipartite component $K_{n,m}$ is planar if and only if $\min\{n, m\} \leq 2$.
Then apply the above lemma and the theorem follows. □



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Conclusion

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- We defined a new graph $\Gamma_C(\mathbb{Z}_n)$ built from complemented elements of $\mathbb{Z}_n$.
- We characterized exactly which elements are complemented: an element must be divisible by the full prime power $p_i^{e_i}$ or not at all.
- We proved $\Gamma_C(\mathbb{Z}_n)$ decomposes as a disjoint union of $2^{k-1}-1$ complete bipartite graphs, where k is the number of distinct prime factors of n.
- The total vertex count is $(2^{k-1}-1)\varphi(n)$.
- We also characterized when the graph is planar.

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- **Other rings:** Can we describe $\Gamma_C(R)$ for other families of rings, like polynomial rings $\mathbb{Z}_n[x]/(f(x))$ or direct products of rings?
- **Relationship to zero-divisor graph:** How does $\Gamma_C(\mathbb{Z}_n)$ sit inside the classical zero-divisor graph $\Gamma(\mathbb{Z}_n)$? It is a subgraph, what structural properties does it inherit or lose?
- **Graph invariants:** Compute the chromatic number, clique number, and independence number for $\Gamma_C(\mathbb{Z}_n)$.
- **Classification:** Classify the family of $\Gamma_C(R)$ for all commutative rings R up to isomorphism.



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References I

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